FLOW PROBLEMS IN MAGNETOHYDRODYNAMICS

(ZADACHI OBTAKANIIA V MAGNITNOV GIDRODINAMIKA)

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We examine the problem of flow around a body from whose interior a magnetic field is excited in the flow of electrically conducting fluid; the solution of this problem is carried out for infinite and large values of the magnetic Reynolds number. The magnetic forces acting on the body are determined. It is shown that for large values of the magnetic Reynolds number the magnetic forces on the body are analogous to viscous friction and profile drag, and can accordingly be called the magnetic friction force and the magnetic profile drag.

We examine the case for which not only a magnetic field but also an electric field is excited from inside the body. It is shown that electric field sources located inside the body have no influence on the flow outside the body.

1. Formulation of the Problem. With the increase in the speed of flying apparatus there is an increase in the ionization of the air flowing through the shock wave ahead of the body. The air flowing around the body becomes electrically conducting, and it becomes possible to influence it by means of a magnetic field.

Let the body, filled with dielectric and containing magnetic poles, be immersed in an electrically conducting fluid. It is necessary to determine the hydrodynamic, magnetic and electrical fields to find the forces acting on the body.

The equations of motion have the form [1, 2]:

$$(\mathbf{V}\cdot\nabla)\mathbf{V} + \frac{1}{\rho}\nabla p = \nu\Delta\mathbf{V} + \frac{1}{4\pi\rho}\operatorname{Rot}\mathbf{H}\times\mathbf{H} \qquad (1.1)$$

$$\operatorname{div}\mathbf{V} = 0, \qquad \operatorname{div}\mathbf{H} = 0, \qquad \operatorname{Rot}(\mathbf{V}\times\mathbf{H}) + \frac{c^2}{4\pi\sigma}\Delta\mathbf{H} = 0$$

Here **V**, **H** are, respectively, the velocity and the magnetic field intensity, p is the pressure, ρ the density, c the speed of light in vacuum, σ the specific electrical conductivity, ν the kinematic coefficient of viscosity. Equations (1.1) are written in the Gaussian system of units. The magnetic permeability of the fluid and the body is taken equal to unity, and ρ, a, ν are assumed to be constant.

For simplicity, incompressible flow is considered. Allowing for compressibility will not introduce any changes in the formulation of the problem if there are no shock waves in the flow. Flows with shock waves are not considered in this paper.

Once equations (1.1) are solved, the intensity of the electrical field **E** in the flow is determined:

$$\mathbf{E} = \frac{c}{4\pi\sigma}\operatorname{Rot}\mathbf{H} - \frac{1}{c}[\mathbf{V}\times\mathbf{H}] \tag{1.2}$$

Inside the body the following equations hold good for the magnetic field:

$$\operatorname{Rot}\mathbf{H} = 0, \quad \operatorname{div}\mathbf{H} = 0 \quad \text{for} \quad \mathbf{H} = -\operatorname{grad}\psi, \quad \Delta\psi = 0 \tag{1.3}$$

where ψ is the scalar potential of the magnetic field. Inside the body the source of excitation of the magnetic field is given, namely, currents flowing in the coils of solenoids. Mathematically, this is equivalent to the specification of the singularities in the solution for ψ.

Assuming that the dielectric constant of the dielectric which fills the body has a constant value, as in (1.3) we have the following equations for the electric field

$$\mathbf{E} = -\operatorname{grad}\varphi; \qquad \Delta\varphi = 0 \tag{1.4}$$

where ψ is the potential of the electric pole.

Next we consider the formulation of the boundary conditions. At infinity the velocity vector and pressure are given, and the intensity of the magnetic and electric fields reduce to zero:

$$\mathbf{V} = \mathbf{V}_\infty, \quad p = p_\infty, \quad \mathbf{H} = 0, \quad \mathbf{E} = 0 \tag{1.5}$$

On the body we have the usual hydrodynamic condition that there shall be no flow through the surface, that is that for non-viscous flow $V_n = 0$ (here the subscript n denotes the velocity vector normal to the surface), or that for viscous flow there shall be no slip, that is, $\mathbf{V} = 0$.

For the component H_n of the magnetic field intensity, on the boundary of the body we have [3] $H_{n2} - H_{n1} = 0$ (index 1 denotes the region inside the body and index 2 denotes the region in the flow). Further, on the body we have

$$\frac{4\pi}{c}\mathbf{i} = \mathbf{n} \times (\mathbf{H}_2 - \mathbf{H}_1) \tag{1.6}$$

where **i** is the surface current density and **n** is the unit external normal to the body.

Let us analyze condition (1.6) in more detail. For this purpose, we return to the last of equations (1.1). Putting this equation in non-dimensional form, we obtain

$$\mathrm{Rot}\,[\mathbf{v} \times \mathbf{h}] + \frac{1}{R_m}\Delta\mathbf{h} = 0 \qquad \left(R_m = \frac{4\pi\sigma V_\infty L}{c^2}\right) \tag{1.7}$$

(The non-dimensional velocity and field intensity are represented by **v** and **h**, respectively.) The dimensionless quantity R_m is usually called the magnetic Reynolds number. In the expression for R_m, the quantities V_∞, the magnitude of the velocity at infinity, and L (a characteristic body length) appear as characteristic quantities. In equation (1.7) the coefficient of the second term characterizes the influence of the dissipation of magnetic energy on the general flow picture. In the limiting case, with R_m infinite, equation (1.7) (returning to dimensional quantities) becomes

$$\mathrm{Rot}\,[\mathbf{V} \times \mathbf{H}] = 0 \tag{1.8}$$

which expresses the conservation of magnetic lines of force for a fluid contour and the absence of dissipation of magnetic energy.

Since on our assumption the body is a dielectric, the surface currents (1.6) flow in an infinitely thin layer of fluid next to the body. If the number R_m is finite, the presence of surface currents leads to an intense dissipation of magnetic energy and a diffusion of the discontinuity; therefore, in stationary flow of an electrically conducting fluid, tangential discontinuities of the magnetic intensity vector are impossible not only in the fluid [2] but also at the wall. Therefore it follows that $\mathbf{i} = 0$ in equation (1.6), so that

$$\mathbf{n} \times (\mathbf{H}_2 - \mathbf{H}_1) = 0 \quad \text{or} \quad \mathbf{H}_{\tau 2} - \mathbf{H}_{\tau 1} = 0 \tag{1.9}$$

where **H** is the component of the magnetic field intensity vector parallel to the wall.

Finally, the condition $H_{n2} - H_{n1} = 0$, and (1.9), for finite R_m give

$$\mathbf{H}_2 - \mathbf{H}_1 = 0 \tag{1.10}$$

For R_m equal to infinity, the conditions on the body remain in the form $H_{n2} - H_{n1} = 0$, together with (1.6), where **i** is to be found from the solution of the problem.

Equation (1.10) may be regarded as a "no-slip" condition between the outer and inner magnetic fields.

Besides the conditions already enumerated, yet another boundary condition has to be satisfied on the surface of the body; this expresses the fact that current cannot flow through the surface. The projection of the vector of the volume density of the current **j** on the normal to the body must be zero:

$$\frac{4\pi}{c} j_n = (\mathrm{Rot}\,\mathbf{H})_{n2} = 0$$

It is not difficult to see that this condition is fulfilled automatically for finite R_m. This follows from equation (1.3) and (1.9). As will be explained below, for R_m equal to infinity this condition is inessential.

After solving equations (1.1) and (1.3) with boundary conditions (1.5), condition $V_n = 0$ (or $\mathbf{V} = 0$ in the presence of viscosity), and condition (1.10) for R_m equal to infinity, equation (1.8) is used instead of the last of equations (1.1), and instead of (1.10), equation (1.6) and the equation of continuity of the normal components of the field are used; the electrical field outside the body can then be found from equation (1.2). To find the field inside the body we have boundary condition [3]

$$\mathbf{E}_{\tau 2} - \mathbf{E}_{\tau 1} = 0 \qquad (1.11)$$

where the index τ denotes the tangential component of the electrical field intensity.

With $\mathbf{E}_{\tau 1}$ found, we determine the electrical field potential ϕ on the body surface; then from Dirichlet's solution of Laplace's equation we find ϕ inside the body.

We may note that on the boundary between the fluid and the body there appear surface charges, whose intensity γ may be found from the relation $4\pi\gamma = E_{n2} - E_{n1}$. From equation (1.2), and allowing for the non-slip condition and the condition that there shall be no flow of current through the body surface ($\mathrm{Rot}\,\mathbf{H}_n = 0$ at the wall), we have $E_{n2} = 0$, and therefore $\gamma = -(1/4\pi)E_{n1}$.

It is evident that the electrical field sources located inside the body, which change the field inside the body, exert no influence on the external flow. In fact, in this case the Laplace equations for the electrical field potential will have to be solved, allowing for the presence of electrical field sources, which is mathematically equivalent to the appearance of certain singularities in the solution. In the final result, only the polarization of the body changes (γ changes).

Next let us determine the forces existing on the body, owing to the flow. The magnetic field will alter the hydrodynamic force. In addition, there appears a magnetic force

$$\mathbf{F}_m = -\iint_s \mathbf{p}_n ds = -\frac{1}{8\pi}\iint_s [H^2\mathbf{n} - 2H_n\mathbf{H}]ds \qquad (1.12)$$

In this integral over the surface s of the body, the stress $\mathbf{p}_n$ is applied to the unit area having the direction of $\mathbf{n}$, the external normal to the body. The vector $\mathbf{p}_n$ with the Maxwell stress tensor for the magnetic field:

$$P_{ik} = \frac{1}{4\pi}\left[\frac{1}{2}H^2\delta_{ik} - H_iH_k\right]$$

If we enclose the magnetic field sources inside the body by a closed surface ω lying entirely inside s, and apply the generalized Ostrogradsky formula to the volume τ between s and $\Im$, we obtain

$$\iint_{s+\omega} \mathbf{p}_n ds = \iiint_\tau \operatorname{div} P d\tau$$

Since $\operatorname{div} P = (1/4\pi)\,\mathbf{H}\times \operatorname{Rot}\mathbf{H} = 0$ everywhere in the volume, in accordance with (1.3), equation (1.12) may be rewritten in the form

$$\mathbf{F}_m = -\frac{1}{8\pi}\iint_\omega [H^2\mathbf{n} - 2H_n\mathbf{H}]\, d\omega$$

Here $\mathbf{n}$ is the outward normal relative to the volume enclosed inside ω.

The magnetic force, formally calculated by integration over the surface of the body according to equation (1.12), thus acts on the magnetic field sources, as was to be expected.

2. Flow around bodies at large values of R_m. In problems of external aerodynamics, even at very high flight speeds, the number R_m is of order unity [4]. However, an investigation of the flow at large values of the number R_m is of considerable interest, for the solutions show how the magnetic and hydrodynamic fluxes change places with increasing values of R_m.

As a preliminary we will examine the problem of $R_m = \infty$. From equation (1.8), which expresses the fact that a magnetic line of force moves with the fluid particles attached to it, and from the condition at infinity $\mathbf{H} = 0$, we obtain $\mathbf{H} = 0$ over the whole flow. From the condition of continuity of the normal component of the field it follows that $H_{n1} = 0$. To calculate the field inside the body it is then necessary to solve equation (1.3) with this boundary condition. Thus, the surface of the body is a surface of tangential discontinuity of the magnetic field. The density of the surface current flowing on the boundary between the body and the fluid is determined by equation (1.6) together with $\mathbf{H}_2 = 0$:

$$\frac{4\pi}{c}\mathbf{i} = -\mathbf{n}\times\mathbf{H}_1$$

(the vector $\mathbf{H}_1$ lies in the plane which is tangent to the body surface). The surface currents shield the magnetic field existing inside the body, so that in the flow we have $\mathbf{H} = 0$.

For example, consider a circular cylinder of infinite length and radius r, along whose axis there flows an electric current of intensity J, in a flow at right angles to its axis. The intensity of the magnetic field on the surface of the cylinder is equal, in absolute value, to $H_1 = 2J/cr$; from this, we find that surface current density, which is constant over the surface, has the absolute value $i = J/2\pi r$; the total surface current is J. Thus we conclude that the net current flowing on the cylinder surface is equal to the given current, and in the opposite direction.

For the case of flow over a body of revolution in whose interior a solenoid is arranged along the axis, the currents flowing on the surface are closed and in a direction opposite to those in the solenoid.

Now let us consider flow at large but finite values of R_m. In this case the tangential discontinuity becomes diffused over a region of thickness $\delta_m = L/\sqrt{R_m}$, where L is a characteristic length [2]. Near the surface of the body there appears a thin layer whose properties are similar to those of the Prandtl boundary layer. The components of velocity and field intensity, normal and tangential to the body surface, are of the order

$$V_n \sim V_0\frac{\delta_m}{L}, \qquad V_\tau \sim V_0, \qquad H_n \sim H_0\frac{\delta_m}{L}, \qquad H_\tau \sim H_0 \tag{2.1}$$

where V_0 and H_0 are a characteristic velocity and magnetic field intensity, respectively. The flow in such layers has been studied in detail by Zhigulev.

Since in practice the case $R_m \sim 1$ is of interest, it is certain that in all cases the magnetic boundary layer is much thicker than the viscous layer. This makes it possible to omit the viscous terms in equations (1.1), putting $\nu = 0$. After solving the equations for the magnetic boundary layer, the problem is solved for the viscous boundary layer, taking the boundary conditions from the magnetic layer problem already solved.

Let us consider the flow in the layer for this flow regime, restricting ourselves to the case of plane flow. We will first make the following observations. From equation (1.2) it follows that the electrical intensity vector is in the direction perpendicular to the plane of the flow, along the z-axis, so from Rot $\mathbf{E} = 0$ we obtain E_z = const, and from the condition at infinity (1.5) it follows that $E_z = 0$.

Therefore the last of equations (1.1), which express the connection between the magnetic and hydrodynamic fields, may be taken in the form of (1.2), with $\mathbf{E} = 0$:

$$[\mathbf{V} \times \mathbf{H}] - \frac{c^2}{4\pi\sigma} \operatorname{Rot} \mathbf{H} = 0 \tag{2.2}$$

Thus, in plane flow there is no polarization of the fluid. This is true also for axisymmetric flows.

With this observation, we now write out equations (1.1) (the last equation is taken in the form (2.2), making the usual boundary layer assumptions based on (2.1):

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + \frac{1}{\rho} \frac{\partial p}{\partial x} - \frac{1}{4\pi\rho} h_y \frac{\partial h_x}{\partial y} = 0, \qquad \frac{1}{8\pi} h_x \frac{\partial h_x}{\partial y} + \frac{\partial p}{\partial y} = 0 \tag{2.3}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \qquad \frac{\partial h_x}{\partial x} + \frac{\partial h_y}{\partial y} = 0, \qquad \frac{\partial h_x}{\partial y} = \frac{4\pi\sigma}{c^2} (v h_x - u h_y)$$

In these equations, x and y are the usual curvilinear orthogonal coordinates for a boundary layer, y is taken normal to the body and x along the curve defining the body, u and v are the velocity components in the directions of x and y respectively, and h_x and h_y are the components of $\mathbf{H}$.

The integral of the second of equations (2.3) is

$$p + \frac{1}{8\pi} h_x^2 = p_0(x) \tag{2.4}$$

where $p_0(x)$ is the pressure on the outer edge of the boundary layer.

The boundary conditions for solving equations (2.3) are taken from the solution of the problem with R_m equal to infinity. On the outer edge of the boundary layer the velocity assumes a given value, determined from the solution of the flow problem in the absence of a magnetic field, and the magnetic field intensity goes to zero:

$$u = u_0(x), \qquad h_x = 0 \qquad \text{for } y \to \infty \tag{2.5}$$

At the wall we have the condition that there shall be no flow through the surface, and we have the value $H_0(x)$ of the tangential component of the magnetic field, here given after being found from the solution of the field equations inside the body for $R_m = \infty$:

$$v = 0, \qquad h_x = H_0(x) \qquad \text{for } y = 0 \tag{2.6}$$

Note that if $h_y = 0$ at the wall for $R_m = \infty$, then for finite R_m this quantity is different from zero, which leads to the appearance of "magnetic drag".

Let us determine the net (hydrodynamic and magnetic) force acting on the body, calculated for unit length of the body. Taking into account the order of magnitude of the terms, putting $H^2 = H_0^2$, $\mathbf{H} = \tau H_0$ in equations (1.12), we obtain

$$\mathbf{F}_m = -\oint \left[p + \frac{H^2(x)}{8\pi}\right] \mathbf{n}\, dx + \frac{1}{4\pi}\oint H_0(x)\, h_y \boldsymbol{\tau} dx$$

Here x is again a length along the curve defining the body, τ is the unit vector in the direction tangential to the body. The integration is in the direction of increasing x.

Using equation (2.4) we have

$$\oint \left(p + \frac{H^2(x)}{8\pi}\right) \mathbf{n}\, dx = \oint p_0(x)\, \mathbf{n}\, dx = 0 \qquad \text{(D'Alembert's Paradox)}$$

Therefore, for the magnetic force we obtain the expression

$$\mathbf{F}_m = \frac{1}{4\pi}\oint H_0(x)\, h_y \boldsymbol{\tau} dx \tag{2.7}$$

The expression in (2.7) is the integral of the tangential forces applied as it were to the body surface, although in fact this force, which may be called the magnetic drag force, is really applied to the magnetic field sources, as noted above.

The force $\mathbf{F}_m$ is directed away from the direction of motion of the body, since magnetic energy is dissipated in the boundary layer and transformed into Joule heat. Therefore, to maintain stationary motion it is necessary to get rid of the energy which is used up in drag. Introducing the dimensionless coefficient of magnetic drag c_{xm}, from equations (2.1) (and allowing for the orders of h_x and h_y) we obtain

$$c_{xm} = \frac{H_0^2}{{}^1/_2\rho_\infty V_\infty^2} \frac{\text{const}}{\sqrt{R_m}}, \qquad c_{xm} = \frac{F_m}{{}^1/_2\rho_\infty V_\infty^2 L} \tag{2.8}$$

The constant in the expression for the magnetic drag depends on the geometry of the body and the geometry of the magnetic source distribution. It will be noted that $c_{xm} \sim R_m^{-1/2}$, by analogy with the case of viscous resistance, where $c_x \sim R^{-1/2}$ when R is the ordinary Reynolds number.

We may also note the possibility that the magnetic boundary layer, which thickens toward the rear of the body, has in turn an effect on the potential flow; this leads to the appearance of a drag which depends on pressure forces. This drag may be called "magnetic profile drag".

BIBLIOGRAPHY

1. Landau, L.D. and Lifshitz, E.M., *Elektrodinamica Sploshnykh sred (Electrodynamics of Continuous Media)*. GTTL, Moscow, 1957.

2. Syrovatskii, S.I., Magnitnaia hydrodinamika (Magnetohydrodynamics). *Usp. Fiz. Nauk* Vol. 12, No. 3, 1957.

3. Tamm, I.E., *Osnov'i teorii electrichestva (Foundations of the Theory of Electricity)*. GTTL, Moscow, 1956.

4. Resler, E.L. and Sears, W.R., The Prospects for Magneto-Aerodynamics. *JAS* Vol. 25, No. 4, 1958.

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